CAN A SOVEREIGN WEALTH FUND IMPROVE FISCAL SUSTAINABILITY?*

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Abstract

We study the macroeconomic implications of a debt-financed sovereign wealth fund (SWF) that invests in the domestic stock market. In a stochastic general equilibrium model with distortionary taxation, we show that the SWF can improve fiscal sustainability despite initially increasing public debt. The SWF generates efficiency gains by reducing distortionary taxation, which stimulates economic growth, but at the cost of amplifying macroeconomic volatility. Paradoxically, this increased volatility benefits the government by deteriorating the hedging properties of stocks, which raises the equity premium that the SWF captures, while simultaneously triggering a flight-to-safety effect that lowers government borrowing costs. These combined effects—higher economic growth, increased stock returns, and reduced interest expenses—enhance fiscal sustainability. The optimal size of the SWF results from balancing efficiency gains against increased risk, with maximum size being optimal under constant marginal efficiency gains of lower taxes. Certain versions of Central Banks' asset purchase programs can be interpreted as a de facto SWF.

Keywords: Balance Sheet Policy, Fiscal Policy, Optimal Policy, Asset Pricing, Portfolio Choice.

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1. Introduction

On February 3rd 2025, the President of the United States signed an Executive Order establishing a Sovereign Wealth Fund (SWF)[1]

> *... to promote fiscal sustainability, lessen the burden of taxes on American families and small businesses, establish economic security for future generations, and promote United States economic and strategic leadership internationally.*

Although SWFs have been around for a while, typically they are used to manage stocks of foreign reserves arising from large trade surpluses or revenues generated by the exploitation of natural resources.[2]Launching such a fund in a country characterized by persistent trade deficits, relatively scarce revenues from natural resources, and already high public debt raises a natural question: How could a US SWF be funded? In this paper, we study the macroeconomic and asset pricing trade-offs of a fully leveraged SWF that issues risk-free liabilities to purchase domestic risky assets.

On the one hand, a fully debt-funded SWF which realizes historical premia of around 6% by investing in the equity market appears attractive. Yet, its implementation raises several questions. What are the costs of accumulating risk on the public balance sheet? Could the associated increase in risk lead to higher borrowing costs that jeopardize the sustainability of public debt? What would be the broader macroeconomic consequences of such an intervention? And finally, given these effects, what would be the correct size of this leveraged SWF?

We address these questions by developing a stochastic general equilibrium framework of a two-period - a short and a long-run- closed endowment economy à la Lucas (1978), featuring two agents and two financial assets -a risky stock and a risk-free bond. The private sector trades in stocks and makes consumption decisions, while the public sector consists of a Treasury and a SWF. The Treasury manages fiscal policy by optimally balancing valuable spending (as in Barro (1981) or Battaglini and Coate (2008)) and distortionary taxation (in the tradition of Barro (1979)), issuing bonds, and has discretionary power. The SWF, operating independently but fiscally backed by the Treasury, issues bonds to purchase stocks.

In this setup, consider a SWF that makes profits on average. The benefits are apparent: the fiscal space is expanded by the transfers and optimal policy prescribes cutting taxes, which de-

[1]The Executive Order can be consulted here.

[2]See James et al. (2022) for a review of commodity-based SWFs.

creases economic distortions, increasing net output, public spending and private consumption. The stimulative effect of the SWF relies critically upon the existence of tax distortions; without them, we recover a version of the Wallace (1981) irrelevance theorem, whereby taxes fall one-to-one with transfers, without affecting output, spending and consumption. This suggests that the benefits of the tax relief are not about lower taxes in itself but about the efficiency gains of lower taxes.

While capturing the equity premium appears attractive for fiscal purposes, it comes at the cost of introducing additional risk into the Treasury's balance sheet. Maintaining a risk-free Treasury portfolio requires the fiscal surplus to adjust to offset the SWF's risk exposure (in line with Jiang et al. (2020)). Under optimal fiscal rules, this adjustment is asymmetric: the Treasury smooths taxes, reducing their sensitivity to output fluctuations, while forcing spending to vary more strongly with economic conditions. As a result, spending becomes more procyclical and taxes less procyclical.[3] This fiscal absorption of aggregate risk spills over to the private sector: in bad states of the world, the asset payoffs are low, the SWF makes losses, the Treasury reduces taxes less than otherwise, further depressing private consumption. Hence, rather than reducing aggregate risk, the SWF amplifies macroeconomic volatility.

If the private sector is aware of these effects, how would it adjust its consumption and portfolio choices in the short run? Since stocks are claims on output, the higher covariance between consumption and output induced by the SWF deteriorates their hedging properties. Through a standard asset pricing mechanism, the increased covariance between the Stochastic Discount Factor (SDF) and stock returns requires a higher equity premium in equilibrium: the SWF makes the stock riskier and induces private investors to require a larger premium for bearing the extra risk. If this effect is strong enough, the SWF crowds out private capital, by decreasing the value of the stocks. This amplified portfolio response stands in contrast to Wallace (1981) neutrality result, where private investors merely offset public portfolio changes.

The reduction in private stock holdings naturally leads to a portfolio rebalancing toward bonds. This flight-to-safety effect increases the demand for government debt, with the strength depending on the level of tax distortions - higher distortions amplify risk transmission through fiscal policy, leading to a stronger reallocation from stocks to bonds. This higher demand for government debt

[3]From a macro-stabilization standpoint, however, one would prefer countercyclical spending and procyclical taxes; thus, a SWF that invests heavily in a claim on aggregate output is unlikely to serve effectively as a providing insurance against aggregate shocks.

is optimally met by higher issuance of bonds. Despite this extra issuance, we find that the risk-free rate falls in equilibrium, as the flight-to-safety demand outweighs the additional bond supply from both the SWF and the Treasury. Moreover, Treasury debt issuance allows to reduce taxes (and smooth them intertemporally), generating efficiency gains that support a higher output, spending and consumption also in the short run.

These results have clear implications for fiscal sustainability. On top of the three conventional determinants of the debt-to-GDP ratio -namely deficits, interest rates and output growth- the SWF introduces a fourth element in the Treasury's intertemporal budget constraint: the capital income derived from operating the SWF (given by the risk premium times the size of the fund). Thus, on top of deficits, the spread between output growth and the interest rate, $g-r$, and between the return on risky asset and the interest rate, $r^s - r$, determine the fiscal capacity. The SWF alters these gaps through two channels. First, by reducing tax distortions, the SWF stimulates g; second, by deteriorating the hedging properties of stocks, the SWF increases r^s and induces a flight-to-safety that lowers r. We conclude that a leveraged SWF may improve debt sustainability. It follows that concerns about the sustainability of existing high debts, for instance those arising from the gap between debt valuations and reasonable measures of expected present value surpluses as in Jiang et al. (2024), might be viewed as an additional motive to pursue a leveraged SWF.

What is the optimal size of a leveraged SWF? If tax distortions are linear, the optimal policy is to fund the SWF to its maximum feasible size, because the constant marginal efficiency gains from tax reductions always outweigh the costs of increased volatility. However, this result disappears with quadratic tax distortions. In this case, the marginal cost from taxation increases with the tax level, and as taxes fall, the marginal benefit from further reductions diminishes. Meanwhile, the costs from higher macroeconomic volatility increase with the size of SWF as government's revenue becomes increasingly dependent on risky capital income. The optimal SWF intervention balances this efficiency-risk trade-off.

In our framework, the SWF can also be interpreted as a central bank that issues reserves to purchase private risky assets—a practice adopted by major central banks globally (e.g., the Federal Reserve purchasing mortgage-backed securities, the Bank of Japan acquiring equities, or the ECB and Bank of England investing in corporate bonds). This perspective uncovers a "Fiscal Channel" of Quantitative Easing (QE) that complements traditional accounts, offering novel insights particularly regarding its effects on risk premia. Beyond the well-documented roles in liquidity provision and yield curve management, this interpretation suggests that through their unconventional monetary policy operations, central banks have effectively functioned as de facto sovereign wealth funds, with potentially important implications for fiscal sustainability.

Related literature. Most of the literature focuses on SWFs financed by revenues from trade of natural resources (see James et al. (2022) for a comprehensive review). We contribute to this literature by studying the macroeconomic and asset pricing implications of a SWF fully financed by public debt. Although the focus of this literature has been on the contribution of SWFs to smooth government spending or hedge aggregate risk (e.g. Tazhibayeva et al. (2008), Sugawara (2014), Van Den Bremer et al. (2016), Koh (2017)), our focus here is on the potential benefits of tax smoothing and risk-taking for the government. In addition, while some papers have emphasized the potential impacts of SWF on financial stability (e.g., Beck and Fidora (2008), Sun and Hesse (2009); Farmer (2017)), or international trade (e.g., Kozack et al. (2010)), we emphasize their effects on the risk premium and fiscal sustainability.

Our theory of the SWF contributes to the debate on fiscal sustainability (see D'Erasmo et al. (2016) and Willems and Zettelmeyer (2022) for comprehensive reviews). While much recent literature has focused on the $r - g$ gap for debt dynamics (e.g., Blanchard (2019), Mehrotra and Sergeyev (2021), Barro (2023)), our SWF framework naturally highlights the importance of the $r^s - r$ spread for debt sustainability. Though this gap has been previously emphasized by Reis (2021) and Reis (2022) as the "Debt Revenue," with an SWF $r^s - r$ times the debt issued represents an explicit revenue stream rather than an implicit rent.[4] Building on the risk management approach to government portfolios pioneered by Bohn (1995) and recently advanced by Jiang et al. (2020), Jiang et al. (2023), and Jiang et al. (2024), we argue that incorporating risk through an SWF can actually improve the government's risk profile.

Our work is also related to a large literature on optimal fiscal policy under incomplete markets. Using a tax deadweight loss function following Barro (1979), we show that the SWF contributes to reducing distortions and smoothing taxes. Similar to Aiyagari et al. (2002), with linear distortions, we find it optimal for the SWF to purchase as many assets as possible to reduce taxes as much as possible. However, the interaction of quadratic distortions with aggregate risk in our model

[4]See Section 6.3 for a comparison between the SWF and Reis' Debt Revenue.

yields an interior solution for SWF asset holdings without requiring the ad-hoc limits present in their framework. Our emphasis on optimal balance sheet structure complements the work by Bohn (1990) and Farhi (2010). While Farhi argues that productivity shocks call for a short position in risky assets, we demonstrate that when government spending is endogenous, as in Barro (1981) and Battaglini and Coate (2008), the opposite result holds because spending serves as an adjustment mechanism. This endogenous spending also distinguishes our research from Jiang et al. (2022), who study optimal fiscal policy with tax distortions but advocate purchasing insurance against aggregate risk and paying a premium. In contrast, we utilize spending as a buffer and highlight the benefits of risk-taking and capturing the premium.

Our analysis also contributes to the debate on the transmission channels of Central Bank's balance sheet policies. While Wallace (1981) established the influential irrelevance result, the modern literature has challenged this neutrality through various mechanisms, including market segmentation (e.g. Vayanos and Vila (2021)), active fiscal policy (e.g., Benigno and Nisticò (2020)), financial frictions (e.g., Gertler and Karadi (2011)), imperfect information and signaling (e.g. Bhattarai et al. (2015), Gaballo and Galli (2024)), nonrational expectations (e.g. Iovino and Sergeyev (2023)), or heterogeneous agents (e.g. Cui and Sterk (2021)). Instead, we break Wallace neutrality through tax distortions, thereby uncovering a distinct fiscal channel of QE that complements the previously identified channels. Our theory closely aligns with the Japanese-style QE studied by Chien et al. (2023) and complements the views of Reis (2017) on how the central bank's balance sheets can expand the fiscal space.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes the equilibrium. Section 4 contains analytical results on the impact of the SWF on the macroeconomy and asset prices. Section 5 endogeneizes the risk-free rate. Section 6 explores the consequences of the SWF for fiscal sustainability. Section 7 discusses the optimal size of the SWF. Section 8 concludes.

2. The model

Consider an economy populated by a continuum of measure 1 of identical households. Time is discrete and the economy only lasts for two periods $t = 0, 1$. This is a stochastic endowment economy, with a Lucas tree that delivers Y_t units of fruits every period, with $Y_t = \mu_Y + \epsilon_t$, where the aggregate shock follows $\epsilon_t \sim \mathcal{N}(0, \sigma_Y^2)$. The amount of fruits represents the only source of risk. Fruits take the form of a single perishable good that delivers utility when consumed. This is the only source of output in the economy.

Assets. There exist two financial assets. Risky assets ("stocks") are denoted by S and represent a claim on output Y_t. They are in fixed supply and marketable at a price P. When the time starts, each investor is endowed with one unit of the stock ($S_{-1}^i = 1$). Besides, there are safe assets ("bonds"), denoted by B, issued in $t = 0$ at a discount price $1/R$ that deliver 1 unit of goods in $t = 1$ with certainty. Both assets are real contracts without default or inflation risk. The economy is closed, so that safe assets are in zero net supply.

The Public Sector. The State consists of two independent agencies. There is a Treasury that chooses a fiscal policy $\mathcal{F} = (\{G_t, T_t\}_{t=0}^1, B^g)$, that is, it chooses spending G and taxes T in both periods and might issue debt B^g in period 0 that it is committed to fully repay in period 1. As in Barro (1979), there is a revenue production function whereby to collect one good, the Treasury must tax $1+H(T)$, with H being a convex function and T the fiscal revenues.[5]This cost function captures in an abstract way deadweight losses associated to taxation due to distorted allocations, collection costs or even unproductive spending. For the analytical discussion, we assume $H(T) = \alpha T$, where α regulates the magnitude of tax losses. The Treasury chooses $\mathcal{F}$ to maximize social welfare. Its budget constraints are given by:

$$\begin{aligned} G_0 &= T_0 + \frac{B^g}{R} \\ G_1 + B^g &= T_1 + X \end{aligned} \tag{1}$$

Furthermore, there is a SWF that chooses a quantity Q of risky assets, and issues a needed debt B^c to purchase them. Any gains or losses that the SWF makes are automatically sent to the Treasury in the form of transfers X. Hence, the SWF bonds are fully back-up by the stream of Treasury's tax revenue. For now, the real interest rate R is a parameter, with the implicit assumption that the central bank commits to a monetary policy that ensures that inflation is well anchored. Section 5 endogeneizes R. The SWF budget constraints are given by:

$$\begin{aligned} QP &= \frac{B^c}{R} \\ B^c + X &= QY_1 \end{aligned} \tag{2}$$

[5]In Barro (1979), H also depends on Y and it is assumed to be homogeneous of degree 1. Tax rate smoothing is the known result. We specify it in terms of revenues, but in the linear case they are equivalent.

The Private Sector. The representative household (which includes all private agents in the economy) makes consumption and portfolio choices $\mathcal{H} = (\{C_t\}_{t=0}^{1}, S, B)$ to maximize its lifetime expected utility. It has Constant Absolute Risk Aversion preferences with the risk aversion level determined by γ. As in Barro (1981), the household derives utility from the consumption of both private and public spending, with the period utility function being $U(C, G) = \omega(-exp\{-\gamma C\}) + (1-\omega)(-exp\{-\gamma G\})$, where ω determines the relative value of private consumption. Her problem can be written as

$$\max_{\mathcal{H}} U(C_0, G_0) + \delta \mathbb{E}_0 \Big[U(C_1, G_1) \Big] \tag{3}$$

where δ is a discount factor. The private sector is subject to these budget constraints

$$\begin{aligned} & C_0 + \frac{B}{R} + PS = (P + Y_0)S_{-1} - T_0 - H(T_0) \\ & C_1 = Y_1 S + B - T_1 - H(T_1) \end{aligned} \tag{4}$$

In period 0, the private sector owns all the stocks, which grants the private sector a claim on total output. This can be viewed as a capital-only economy, where stocks represent ownership of the total output generated by the underlying capital. In addition, the private sector pays taxes, including the deadweight losses of the tax. With the after-tax resources, the sector decides on consumption, bonds and stocks.

Timing. The timing protocol is as follows. First, the SWF announces its policy Q. Given that policy, both the private sector and the Treasury make their choices $\mathcal{H}, \mathcal{F}$, respectively. In period 1, the Treasury has discretionary power to adjust its policies conditional on the short-run competitive equilibrium and the output realization. Given the discretionary period 1 fiscal policy, agents make their decisions in the short-run. The short- and long-run competitive equilibria are indexed by Q. The SWF chooses Q to deliver the best competitive equilibrium.

3. Equilibrium

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3.1. The Long-Run Equilibrium

The long-run competitive equilibrium consists of a list of allocations $\{Y_1^*, C_1^*, G_1^*, T_1^*\}$ that satisfy market clearing conditions and optimal rules for the private and public sector. Since this is the last period, asset markets do not open, so there are no allocations or prices for assets to determine.

3.1.1 The Treasury

In period 1, after observing output, the Treasury chooses taxes and spending to maximize social welfare

$$\max_{\{T_1, G_1\}} U(C_1, G_1) \tag{5}$$

constrained by the allocations and prices emerging from the short-run competitive equilibrium, the budget constraints for him, the SWF and the private sector, and market clearing conditions. The Treasury ignores any predetermined plans for taxes and spending it might had in advance, and chooses what is best in the realized state of the world.

The solution to the Treasury problem are two fiscal rules contingent on the short-run competitive equilibrium and Q. Operating on the First Order Conditions, the tax rule reads

$$T_1^* = \frac{1}{2+\alpha}\Bigg(Y_1 + 2B^g - a - 2X\Bigg) \tag{6}$$

with $a \equiv \frac{ln[\omega(1+\alpha)] - ln[(1-\omega)]}{\gamma}$. Realized transfers are given by

$$X = QP\Big(\frac{Y_1}{P} - R\Big) \tag{7}$$

That is, transfers are equal to the total assets bought by the SWF (QP) times the earned premium, which is the spread between the realized risky return the SWF gets minus the safe return paid on bonds $\Big(\frac{Y_1}{P} - R\Big)$. In turn, the equilibrium spending rule is

$$G_1^* = \frac{1}{2+\alpha}\Bigg(Y_1 - \alpha B^g - a + \alpha X\Bigg) \tag{8}$$

Tax revenues are procyclical and respond to the stock of debt carried from period 0, as it needs to be honored. Spending is also procyclical, but is inversely related to public debt, as it competes with it for limited resources. Moreover, both policy rules include a term related to SWF's transfers which, if positive, allows simultaneously for both lower taxes and higher spending. Thus, fiscal rules are state-contingent: in good times (high Y_1, $X > 0$), the Treasury taxes and spends more. Note these rules are the optimal behavior of a government without commitment, in contrast with the ad-hoc rules commonly used in the fiscal-monetary interactions literature.

3.1.2 The Private Sector

In the long-run, the private sector consumes all the resources it obtains from its short-run savings after paying taxes. In equilibrium, private consumption is a residual: the net output that the public sector does not consume. Net output is given by the exogenous gross output minus the loss of resources associated to taxation:

$$Y_1^* \equiv Y_1 - H(T_1) = \frac{2}{2+\alpha}\left(\alpha a/2 + Y_1 - \alpha B^g + \alpha X\right) \tag{9}$$

It is increasing on gross output and SWF's transfers, as higher transfers reduce tax distortions, but decreases with public debt, as more debt necessitates higher taxes for repayment. It follows that the equilibrium private consumption reads as

$$C_1^* = Y_1^* - G_1^* = \frac{1}{2+\alpha}\left((1+\alpha)a + Y_1 - \alpha B^g + \alpha X\right) \tag{10}$$

3.2 The Short Run Equilibrium

In period 0, agents make decisions anticipating the long-term equilibrium outcomes and taking $\mathcal{W}$ as given. We start by solving the problems of the Treasury and the private sector.

3.2.1 The Treasury

In the short run, the Treasury faces the same intra-temporal trade-off between taxes and spending as in period 1. In addition, the Treasury can issue bonds. While bond issuance introduces an intertemporal dimension, we assume that the Treasury supplies bonds on demand. The reason is that R is a parameter and the SWF's bond issuance B^c is taken as given. Under these conditions, the bond market can clear through two channels: (1) a form of financial repression, where the private sector is forced to hold the combined bond supply $B^c + B^g$; (2) allowing private bond demand to adjust freely at the prevailing R, with the Treasury accommodating excess demand or supply. We focus on the second mechanism, where Treasurys face demand constraints on bond issuance. Although we could formulate the Treasury's problem dynamically with an explicit bond market clearing constraint, this would make the clearing condition rather than the intertemporal Euler equation determine the equilibrium debt level. The following formulation simplifies the exposition by simply focusing on the intra-temporal problem:

$$\max_{\{T_0, G_0, B^g\}} U(C_0, G_0)$$

The opportunity set is constrained by its budget, the private sector budget, and short-run market clearing conditions. Using the FOC for taxes, the SWF budget constraint and the bonds market clearing condition, the tax rule becomes

$$T_0 = \frac{1}{2+\alpha}\Big(Y_0 - 2\frac{B^g}{R} - a\Big) \tag{11}$$

Taxes increase with output and decrease with debt, highlighting the two main avenues for raising Treasury revenue: distortionary taxation and debt. This trade-off reflects the balance between immediate revenue generation and future financial obligations. Using the bond market clearing condition, $B^g = B - B^c$, that is, the Treasury accommodates excess demand or supply in the bond market.

3.2.2 The Private Sector

The optimality condition that characterizes the optimal inter-temporal allocation for bonds is

$$exp\{-\gamma C_0\} = \delta R \mathbb{E}_0\Big[exp\{-\gamma C_1\}\Big] \tag{12}$$

This condition alongside the probability density function for future consumption, which agents know, can be used to derive a consumption function:

$$C_0 = \mu_C - \frac{\gamma \sigma_C^2}{2} - \frac{1}{\gamma}(\tilde{r} - \rho) \tag{13}$$

where $\tilde{r} \equiv lnR$ and $\rho \equiv -ln\delta$. This demand function captures two key effects: consumption smoothing and precautionary savings. When future consumption is expected to be high, households save less and consume more in the present (smoothing). Conversely, if future consumption is anticipated to be volatile, individuals self-insure by reducing current consumption, with the extent of this adjustment depending on their level of risk aversion (precautionary savings). The effect of Q on C_0 is contradictory: while Q increases expected future consumption, encouraging higher current consumption, it also increases consumption variance, prompting individuals to self-insure against potential adverse outcomes.

The optimality condition that characterizes the optimal inter-temporal allocation for stocks is

$$exp\{-\gamma C_0\} = \delta \frac{1}{P} \mathbb{E}_0 \Big[exp\{-\gamma C_1\} Y_1 \Big] \tag{14}$$

3.3 Competitive Equilibrium Definition

Given Q, and realized gross output $\{Y_t\}$, a competitive equilibrium is a set of allocations $\{C_t^*, B^*, S^*, (B^g)^*, T_t^*, G_t^*, ($ and the risky price P^* such that the following conditions are satisfied:

1) Short and long-run consumption rules (13)-(10).

2) Stock's Euler equation (14).

3) Tax and spending rules (11)-(6)-(8).

4) The budget constraints for the private sector, public sector, and central bank.

5) Market clearing conditions for bonds $B = B^g + B^c$, stocks $Q + S = 1$, and goods $C_t + G_t = Y_t - H(T_t)_{t=0,1}$.

Expressions (6)-(8)-(9)-(10) characterize the equilibrium in period 1. Given that, in the short run, there are 8 equations and 8 unknowns. The balancing mechanisms are as follows. The stock price P balances the stock market; the Treasury balances the bond market by supplying bonds on demand. The private budget constraint is balanced by the demand of bonds; the Treasury budget constraint is balanced by spending. The Gtwo last constraints imply the clearing of the goods market. The competitive equilibrium can be solved in closed-form; the equilibrium variables are used in the next sections to analyze the effects of the SWF.

4. The Effects of the Sovereign Wealth Fund

This section analyzes the effects of the asset purchases by the SWF on the macroeconomic and financial variables. We focus on the response of the variables in the margin, which we call "Q-multipliers", and on the risk absorption.

4.1. Risk absorption by fiscal means

While capturing the expected excess return is appealing for the Treasury, it comes at a cost: the public balance sheet becomes riskier. The key to understanding the macroeconomic implications of a SWF lies in examining how this additional risk is absorbed by fiscal policy—and ultimately, how it spills over to the private sector.

Result 1. Aggregate Risk Absorption with a Sovereign Wealth Fund.

i) The fiscal surplus absorbs the SWF risks. State-contingent transfers from the SWF X introduce extra risk into the Treasury's budget constraint. To maintain a risk-free (zero-beta) Treasury debt portfolio, the fiscal surplus must precisely offset this new exposure. Formally, the surplus beta satisfies:

$$\beta_{T_1-G_1} = -\beta_X = -Q \tag{15}$$

where $\beta_W = \frac{Cov(W^*,Y_1)}{Var(Y_1)}$ for any variable W.

ii) Asymmetric risk absorption. The extra risk is absorbed by spending while taxes are smoothed out:

$$\frac{d\beta_{T_1}}{dQ} = -\frac{2}{2+\alpha} < 0, \quad \frac{d\beta_{G_1}}{dQ} = \frac{\alpha}{2+\alpha} > 0 \tag{16}$$

iv) Private consumption becomes riskier

$$\frac{d\beta_{C_1}}{dQ} = \frac{\alpha}{2+\alpha} \tag{17}$$

Proof. Appendix 1.

The result is stated in terms of "output-betas"; Appendix 1 shows that these betas are proportional to "risk-betas", where the covariance is measured with respect to the SDF $e^{-\gamma(C_1^*-C_0)}$.

As the public sector invests in risky assets, maintaining a zero-beta (i.e., risk-free) Treasury debt portfolio demands that the fiscal surplus adjust to offset the SWF's new risk exposure (as in Jiang et al. (2020)). Under optimal fiscal rules, this adjustment is asymmetric: the Treasury chooses to smooth taxes—reducing their sensitivity to output fluctuations—while allowing spending to vary more strongly with economic conditions. As a result, spending becomes more procyclical and taxes less procyclical. Or, equivalently, the SWF increases taxpayers insurance against aggregate risk by cutting spending more in bad times.[6] From a macro-stabilization standpoint, however, one would prefer countercyclical spending and procyclical taxes; thus, a SWF that invests heavily in a claim on aggregate output is unlikely to serve effectively as a stabilization mechanism.

Without tax distortions, spending does not react to Q and then $\frac{\partial \beta_{T_1}}{\partial Q} = -\frac{\partial \beta_X}{\partial Q} = -1$. In this case, taxes become more detached from output. When raising taxes is more expensive (higher α), the Treasury has a greater incentive to smooth taxes rather than let them fluctuate with output, leaving spending as the main absorber of macro risk.

The digestion of aggregate risk through fiscal means has implications for private consumption. As in equilibrium private consumption is related to output and spending ($C_1^* = Y_1^* - G_1^*$), it turns out that consumption becomes riskier, as shown by the increase in beta. An alternative measure of consumption riskiness is:

$$\frac{\partial \sigma^2_{C^*}}{\partial Q} = \left(\frac{1+\alpha Q}{2+\alpha}\right)\frac{4\alpha}{2+\alpha}\sigma^2 = \frac{\partial \sigma^2_{G^*}}{\partial Q}$$

The mechanics behind this risk pass-through from the Treasury's balance sheet to the private sector is as follows. In bad states of the world, the asset's payoff is low, the SWF incurs losses, the Treasury reduces taxes ($\beta_{T_1} > 0$) but less than otherwise ($\frac{\partial \beta_{T_1}}{\partial Q} < 0$), further depressing private consumption when it is already weak. Without tax distortions, all the adjustment goes through taxes $\frac{\partial \beta_{T_1}}{\partial Q} = -1$, that fall much more in bad states, insuring taxpayers.

Altogether, rather than making risk magically disappear on the Treasury's balance sheet, the SWF ends up generating additional macroeconomic volatility. This amplification of the risk borne by the private sector through fiscal policy have important consequences for asset prices and portfolio allocations in the short-run.

[6] In the model, taxpayers and spending-users are the same individuals, therefore there is no insurance but one can think of more general situations where the two groups differ.

4.2. Financial markets response

This section analyzes the implications for asset pricing and portfolio choice of the increase in private consumption risk due to the SWF. Result 2 collects the main effects.

Result 2. The effect of a Sovereign Wealth Fund on financial markets. *An increase in Q leads to i) Lower stock prices*

$$\frac{dP^*}{dQ} < 0$$

ii) A higher risk premium

$$\frac{d\mathbb{E}_0(Y_1/P^* - R)}{dQ} > 0$$

ii) A strong fall in the private stock demand

$$\frac{\partial S^d}{\partial Q} = -\frac{2+2\alpha}{2+\alpha} < -1$$

and, if $\mu_Y - PR \geq \gamma\sigma_Y^2\frac{1+\alpha Q}{2+\alpha}$, a strong increase in the private bond demand:

$$\frac{\partial}{\partial Q}\frac{B^d}{R} = \frac{\alpha(\mu_Y - PR)}{(2+\alpha)} - \frac{\alpha\gamma\sigma_Y^2(1+\alpha Q)}{(2+\alpha)^2} + \frac{(2+2\alpha)P}{2+\alpha} > \frac{\partial}{\partial Q}\frac{B^c}{R} = P$$

Proof. Appendix 2.

In the proof, we show that the equilibrium pricing function is given by:

$$P^* = \frac{\mu_Y - \gamma\left(\frac{1+\alpha Q}{2+\alpha}\right)\sigma_Y^2}{R} \tag{18}$$

The equilibrium function depends on two elements: (1) expected payoffs discounted at the risk-free rate; (2) a risk adjustment component that brings the risky price down, reflecting the premium investors require for bearing risk. The risk adjustment depends on the physical risk $\left(\frac{1+\alpha Q}{2+\alpha}\right)\sigma_Y^2$ times the risk aversion γ. Since $\alpha \geq 0$, a higher Q leads to a higher risk adjustment, depressing the equilibrium asset price. To ensure positive prices, $\mu_Y > \gamma\left(\frac{1+\alpha Q}{2+\alpha}\right)\sigma_Y^2$ is assumed throughout.

These effects flow from a common source: the SWF's impact on aggregate risk distribution. As shown in *Result 1*, the SWF makes private consumption more procyclical, increasing its covariance with output. Since stocks are claims on output, this higher covariance means they provide less insurance against consumption risk - they pay off less precisely when marginal utility is high. Through the pricing kernel, risk-averse investors demand higher compensation for holding stocks, raising the equity premium. This deterioration in stocks' hedging properties triggers substantial portfolio rebalancing. Private investors reduce their equity exposure by more than the SWF's purchases, and redirect these funds toward safe government bonds.

4.3 Multipliers

We now analyze the equilibrium effects of the Sovereign Wealth Fund intervention on the macroeconomy. Using the closed-form solution of the equilibrium derived above, we characterize the sign and magnitude of these effects through a series of multipliers.

Result 3. The Effect of a Sovereign Wealth Fund on the Macroeconomy.

i) *The short-run multiplier on taxes is:*

$$\frac{dT_0^*}{dQ} = -\frac{2\alpha\gamma\sigma_Y^2}{(1+R)(2+\alpha)^2}Q$$

and in the long run:

$$\frac{dT_1^*}{dQ} = -\frac{2}{2+\alpha}\left[Y_1 - \mu_Y + \gamma\frac{1+\alpha Q}{2+\alpha}\sigma_Y^2 + \frac{\alpha Q\gamma\sigma_Y^2}{(2+\alpha)(1+R)}\right]$$

ii) *The multiplier on output is proportional to the reduction in tax distortions:*

$$\frac{dY_t^*}{dQ} = -\alpha\frac{dT_t^*}{dQ}$$

iii) *The multiplier on Treasury spending and private consumption is identical and equals half that of net output:*

$$\frac{dG_t^*}{dQ} = \frac{dC_t^*}{dQ} = \frac{1}{2}\frac{dY_t^*}{dQ} = -\frac{\alpha}{2}\frac{dT_t^*}{dQ}$$

iv) *The multiplier on Treasury debt is:*

$$\frac{d(B^g)^*}{dQ} = \frac{RQ\alpha\gamma\sigma_Y^2}{(1+R)(2+\alpha)}$$

v) *The multiplier on private bond holdings is:*

$$\frac{dB^*}{dQ} = \mu_Y - \frac{\gamma\sigma_Y^2(1+2\alpha Q + R(1+\alpha Q))}{(1+R)(2+\alpha)}$$

vi) *The multiplier on the SWF bonds is:*

$$\frac{d(B^c)^*}{dQ} = \mu_Y - \frac{\gamma\sigma_Y^2(1+2\alpha Q)}{(2+\alpha)}$$

Proof: These multipliers are derived from the system of equations characterizing the competitive equilibrium by taking the total derivative of each variable with respect to Q.

Result 3 reveals that SWF asset purchases reduce taxes in the short run, with the magnitude depending positively on the level of tax distortion, risk aversion, and aggregate risk. The effect intensifies with the size of the purchases, indicating that larger SWF interventions yield greater tax relief. In the long run, the multiplier is contingent on output realization; when output exceeds its mean, the SWF intervention leads to substantial tax reductions. Notably, even when output falls below its mean, tax reductions can still materialize if aggregate risk—and consequently the risk premium—is sufficiently high. The tax reductions generate static effects in both periods: lower taxes alleviate distortions and stimulate output proportionally to the tax distortion parameter. The resulting additional resources are equally divided between government spending and private consumption.

On the debt side, SWF purchases increase the debt issued by the SWF, but by less than partial equilibrium analysis would suggest. This occurs because SWF purchases reduce asset prices, making each unit of Q less expensive and requiring relatively less debt issuance. The private sector's response to changes in bond demand has an ambiguous sign: higher expected output increases stock prices and SWF debt issuance, which the private sector absorbs, but there is also a risk adjustment term that grows with Q. Nevertheless, the private sector ultimately demands more bonds than the SWF supplies; this excess demand is satisfied by Treasury issuance, representing a flight-to-safety effect that increases monotonically with distortions and risk.

The mechanism underlying these equilibrium multipliers operates as follows. First, the SWF intervention deteriorates the hedging properties of risky assets, leading to a significant decline in private demand for these assets—a decline that more than offsets the additional demand generated by the central bank—and triggers a flight-to-safety that boosts demand for Treasury bonds. The Treasury meets this demand by increasing bond issuance, which enables it to reduce short-run taxes. The consequent alleviation of distortions enhances output, supporting higher levels of both private and public consumption. In the long run, the SWF provides additional revenue on average, permitting lower taxes, higher net output, and increased private and public consumption.

The SWF multipliers increase proportionally with the magnitude of tax distortions. Greater distortions amplify the impact of tax reductions on net output, thereby enhancing consumption and spending. This result underscores the potential benefits of shifting public revenue generation away from distortionary taxation toward capital income generated through the SWF. The following results highlights the adjustment in the special case of no distortions.

Result 4. Irrelevance of a Sovereign Wealth Fund.

Wallace (1981)'s irrelevance result emerges as a special case when $\alpha = 0$. Under these conditions:

i) *Long-run taxes bear all the adjustment while short-run taxes remain unchanged:*

$$\frac{dT_1^*}{dQ} = -(Y_1 - \mu_Y + \frac{\gamma}{2}\sigma_Y^2) = -\frac{dX^*}{dQ}; \quad \frac{dT_0^*}{dQ} = 0 \tag{19}$$

ii) *The decline in private stock demand perfectly offsets the additional demand from the SWF, and bond demand exactly absorbs the SWF issuance:*

$$\frac{dS^d}{dQ} = -1; \quad \frac{dB^d}{dQ} = PR \tag{20}$$

iii) *Asset prices, consumption, output, and spending remain unchanged:*

$$\frac{dP^*}{dQ} = \frac{dY_t^*}{dQ} = \frac{dC_t^*}{dQ} = \frac{dG_t^*}{dQ} = 0 \tag{21}$$

Proof. This follows as a corollary from the multipliers in Result 3.

In the absence of distortions, Q has no impact on real variables and simply entails an issuance of debt by the SWF that is absorbed one-to-one by the private sector. Thus, our model nests Wallace (1981)'s irrelevance theorem as a special case. In a distortion-free economy, the Treasury would reduce taxes one-to-one with transfers without affecting spending, consumption, or output since all gains/losses are transferred back to the private sector without distorting decision margins, preserving the original resource allocation and rendering the SWF intervention ineffective.

5. Equilibrium with an Endogenous Interest Rate

While the previous setup allows the analytical characterization of the novel channels at work, an important assumption was an exogenously given risk-free rate. The expansion of deficits and the accumulation of debts in the public balance sheet were absorbed by quantity adjustments. This section departs from that assumption, studying price adjustments instead.

Consider an alternative closure for the bond market: all agents, including the Treasury, optimally decide how much debt to issue in period 0, and the risk-free rate R adjusts to balance the market. For the Treasury, this setup entails a determination of the debt issuance according to the following Euler equation

$$U_{C_0} - U_{G_0} = \delta R \mathbb{E}_0 \left[U_{C_1^*} \frac{\partial C_1^*}{\partial B^g} + U_{G_1^*} \frac{\partial G_1^*}{\partial B^g} \right] \tag{22}$$

The equation balances the short-run marginal costs of issuing debt instead of taxing agents in period 0, against the future marginal gains of higher consumption (from the bond interests) but lower spending.

Figure 1 presents the equilibrium variables as a function of the SWF intervention Q, comparing outcomes under low and high levels of aggregate risk. The results align with the previous analytical results in several dimensions. Taxes decrease, while both Treasury spending and private consumption increase monotonically with Q, regardless of the level of aggregate risk. These effects remain robust to changes in uncertainty, confirming that the SWF's intervention alleviates the fiscal burden and raises private sector consumption by redistributing the returns from financial markets.

However, this alternative closure of the bond market has significant effects on financial market variables. The most significant change emerges in the behavior of the risk-free rate R, which declines

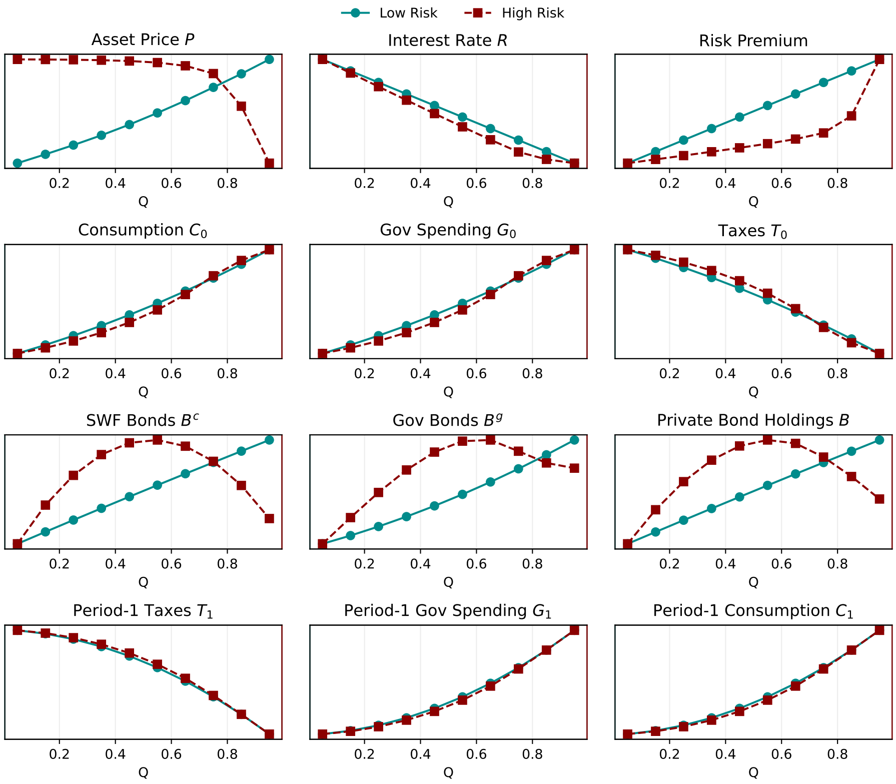

Figure 1: Equilibrium variables as a function of Q with endogenous risk-free rate. *The graph uses different y-axis for low and high risk. The calibration is:* $(\alpha, \gamma, \omega, \delta, \mu_Y, Y_0, \sigma_Y^{low}, \sigma_Y^{high}) = (2, 2, 0.6, 0.95, 1, 0.8, 0.2, 0.8)$.

as Q increases, signaling a strong flight-to-safety effect: the rise in demand for bonds—driven by a deterioration in the stock's hedging properties—more than offsets the additional bond supply from both the SWF and the Treasury. Under low aggregate risk, this decline in R leads to an increase in stock prices as Q rises. Whereas previously Q only exerted downward pressure on P through risk adjustments, it now also influences P via the interest rate channel in the other direction. The prior negative relationship between Q and P can be recovered with a sufficiently high risk level such that the higher risk outweighs the lower discount. In such high-risk regime, as P decreases the SWF can finance its interventions with lower debt issuance, allowing for a more extensive stock purchase program with reduced leverage. These findings underscore the pivotal role of aggregate risk in shaping the transmission of SWF interventions through financial markets.

6. Fiscal Capacity with a Sovereign Wealth Fund

The preceding analysis illustrates the short- and long-run effects of a sovereign wealth fund through its spillovers on asset and goods markets. Given that one of the primary motivations for establishing an SWF is to improve the fiscal outlook, an important question is whether and how the SWF indeed enhances fiscal capacity. This section reorganizes the previous results from this standpoint.

6.1. The Intertemporal Budget Constraint

The long-run and short-run effects described above can be integrated via the Treasury's intertemporal budget constraint (IBC). By consolidating the two-period budget constraints and taking conditional expectations, we obtain this accounting dynamic equation:

$$\frac{B^g_{-1}}{Y_0} = \frac{T_0 - G_0}{Y_0} + \mathbb{E}_0\left[\frac{1+g}{1+r}\frac{T_1 - G_1}{Y_1}\right] + \frac{QP}{Y_0}\mathbb{E}_0\left[\frac{Y_1/P - R}{1+r}\right], \tag{23}$$

where $g = Y_1/Y_0 - 1$ and $r = R - 1$. In the standard formulation, the market value of Treasury debt equals the expected present value of future fiscal surpluses. When variables are normalized by output, the discount factor incorporates output growth, implying that higher growth supports a higher level of debt for a given surplus or, equivalently, a larger deficit for a given debt stock. If initial debt is zero ($B^g_{-1} = 0$), as assumed previously, the expression indicates that current deficits must be offset by appropriately discounted expected future surpluses. The SWF introduces an additional term: the discounted profits earn from the SWF operations, which equals the product the risk premium times the size of the SWF. Thus, rather than relying solely on future surpluses, the Treasury may leverage SWF profits to bolster its fiscal stance, thereby expanding the fiscal space in a nontraditional manner.

Although the debt arithmetics are useful to highlight the appearance of a new term, whether or not this extra term expands the space once the other variables are allowed to react requires further analysis. For example, the extra risk in the public balance sheet might increase interest rates, reducing or even reversing the equity premium and the anticipated gains. Our model serves to elucidate these general equilibrium concerns. To highlight the relevant variables, define the normalized deficit-to-output ratio as $\mathcal{S}^*_t \equiv \frac{T^*_t - G^*_t}{Y^*_t}$, the normalized initial Treasury debt as $(\mathcal{B}^g_{-1})^* \equiv \frac{B^g_{-1}}{Y^*_0}$, the normalized SWF debt as $(\mathcal{B}^c)^* \equiv \frac{(B^c)^*}{Y^*_0}$. Plugging the equilibrium relationships into the

IBC, one obtains

$$(\mathcal{B}^g_{-1})^* = \mathcal{S}^*_0 + \mathbb{E}_0\left[\frac{1+g^*}{1+r}\mathcal{S}^*_1\right] \qquad \underbrace{+\frac{(\mathcal{B}^c)^*}{1+r}\mathbb{E}_0\left[(r^s)^* - r\right]}_{\text{Extra revenue from the SWF}}, \tag{24}$$

where $(r^s)^* = Y_1/P^* - 1$. This is a dynamic accouTnting identity with the equilibrium relationships incorporated. The expression shows that, in addition to the fiscal surpluses, the gaps $(g-r)$ and $(r^s - r)$ are crucial determinants of the fiscal space. The following result collects the effects of the SWF on the fiscal space.

Result 5A. The Sovereign Wealth Fund and the Fiscal Space. *Debt-financed stock purchases enhance the fiscal capacity. For any given level of initial debt B^g_{-1}, the SWF supports larger deficits in both the short and long run:*

$$\frac{d\mathcal{S}^*_t}{dQ} < 0, \quad t = 0, 1,$$

provided that deficits satisfy $G_t - T_t < Y_t\frac{(1+\alpha/2)}{\alpha}$. Larger deficits are supported by a higher expected growth - yield gap

$$\frac{d\mathbb{E}_0[g^* - r]}{dQ} > 0,$$

and higher SWF profits

$$\frac{d\frac{(\mathcal{B}^c)^*}{1+r}\mathbb{E}_0[(r^s)^* - r]}{dQ} > 0$$

if $\left(1 - \frac{2\alpha^2\gamma\sigma^2_Y Q^2}{(1+R)(2+\alpha)^2}\right) > 0$.

Proof. Appendix 3.

Result 5 shows that the SWF induces an expansion of the fiscal space by widening the gaps $(g-r)$ and (r^s-r), allowing larger deficits for a given level of initial debt. The condition $\left(1-\frac{2\alpha^2\gamma\sigma^2_Y Q^2}{(1+R)(2+\alpha)^2}\right) > 0$ ensures the SWF debt-to-output ratio increases with Q. While this condition is easily satisfied for moderate values of Q, it is likely to be violated as Q approaches 1 for reasonable parameter values. In such cases, debt issuance by the SWF actually reduces its debt-to-output ratio. Although this may seem appealing at first glance, it has the disadvantage of capturing smaller profits.

The result is established analytically for an economy with an exogenous R. Appendix 5 also shows that although the extra growth helps, it is not enough to compensate for the larger period 1 deficits, that is $\frac{d}{dQ}\mathbb{E}_0[(1+g^*)\mathcal{S}_1^*] < 0$; the revenues from the SWF are then crucial for the expansion of the fiscal space. When R is endogenous, the effect is further reinforced since R is monotonically decreasing in Q at a faster pace than stock prices increase (as shown in Section 5), widening further the $(g-r)$ and $(r^s - r)$ gaps.

An alternative interpretation of *Result 6* is as follows. If the initial debt B_{-1}^g is excessively high relative to the expected path of fiscal surpluses—implying an unsustainable debt trajectory—the SWF can offset this imbalance. Although paradoxical at first sight, issuing more debt to fund the SWF can actually improve the sustainability of outstanding total public debt. A high level of initial debt-to-output might not be an excuse but rather a motive to create a SWF. Specifically, if the initial debt-to-output ratio $\mathcal{B}_{-1}^g$ exceeds the present value of expected surpluses, concerns about sustainability might arise. However, through debt issuance and the accrual of a risk premium, the SWF can sustain the debt without requiring fiscal adjustment. In particular, the requisite size of the SWF is determined by the gap between the outstanding debt and the expected present value of surpluses, scaled by the inverse of the risk premium:

$$\begin{aligned}
\mathcal{B}^c &= \frac{1+r}{\mathbb{E}_0\left[r^s - r\right]} \times \left(\mathcal{B}_{-1}^g - \mathbb{E}_0\left[\mathcal{S}_0 + \frac{1+g}{1+r}\mathcal{S}_1\right]\right) \\
&\approx \underbrace{(1-\mathbb{E}_0[r^s])}_{\text{Damper}} \times \underbrace{\left(\mathcal{B}_{-1}^g - \mathbb{E}_0\left[\mathcal{S}_0 + \frac{1+g}{1+r}\mathcal{S}_1\right]\right)}_{\text{Debt Valuation Gap}}
\end{aligned} \tag{25}$$

For each unit of the valuation gap, the SWF must issue debt equal to the inverse of the equity premium. To illustrate, consider a present value of deficits equal to 1% of GDP. In the conventional framework, this corresponds to the issuing 1% of GDP in new debt; however, the SWF mechanism reduces the required issuance of SWF debt to about 0. 94% of GDP, thus mitigating the overall debt burden. In equilibrium, the SWF further amplifies the $(g-r)$ and $(r^s - r)$ gaps, reinforcing this arithmetic derivation. If the SWF appears to be beneficial for debt sustainability, should it be as big as possible? Section 7 examines this issue.

6.2. The valuation approach

Some articles in the literature read the IBC from a valuation point of view, replacing the Treasury rate r by the SDF to discount future surpluses (e.g. Bohn (1995) or Jiang et al. (2024)). Then, the IBC ceases to be an accounting identity and becomes a valuation equation. Since the SWF brings risks to the government balance sheet In our model, the equilibrium SDF is given by $M^* = \delta\frac{exp\{-\gamma C_1^*\}}{exp\{-\gamma C_0^*\}} = \frac{1}{1+m^*}$. In this case, the IBC becomes

$$(\mathcal{B}^g_{-1})^* = \mathcal{S}_0^* + \mathbb{E}_0\left[\frac{1+g^*}{1+m^*}\mathcal{S}_1^*\right] \quad + \frac{QP^*}{Y_0^*}\mathbb{E}_0\left[\frac{(r^s)^*-r}{1+m^*}\right] \tag{26}$$

Thus, the market value of the outstanding government bond portfolio equals the present risk-adjusted discounted value of current and future primary surpluses plus the risk-adjusted discounted profits from the SWF. Opening the expectation operators,

$$\mathbb{E}_0\left[\frac{1+g^*}{1+m^*}\mathcal{S}_1^*\right] = \mathrm{Cov}_0(M^*, T_1^*) - \mathrm{Cov}_0(M^*, G_1^*) + \mathbb{E}_0\left[\frac{1}{1+r}\mathcal{S}_1^*\right]$$

and

$$\mathbb{E}_0\left[\frac{(r^s)^*-r}{1+m^*}\right] = \mathrm{Cov}_0(M^*, (r^s)^*) - \mathrm{Cov}_0(M^*, r) + \mathbb{E}_0\left[\frac{(r^s)^*-r}{1+r}\right]$$

Hence, the valuation approach boils down to the IBC plus a risk correction (Jiang et al. (2023)):

$$(\mathcal{B}^g_{-1})^* = \mathcal{S}_0^* + \mathbb{E}_0\left[\frac{1+g^*}{1+r}\mathcal{S}_1^*\right] \ + \frac{(\mathcal{B}^c)^*}{1+r}\mathbb{E}_0\left[(r^s)^*-r\right] + \mathcal{R}_0 \tag{27}$$

where $\mathcal{R}_0$ collects all the covariances. Appendix 6 demonstrates that the sign of the covariance is determined by the slope measuring each variable's sensitivity to Y_1. This covariance is generally negative for G_1 and r^s, and if $Q < 1/2$, also for taxes. The intuition for this negative sign is straightforward: since the variables are affine in output, in good times, consumption, taxes, spending and returns go up, but marginal utility goes down. The negative sign for taxes and returns reduces the fiscal space; the negative sign for spending expands the space.

If the SWF puts more risk on the balance sheet...wouldn't that risk simply appear on $\mathcal{R}_0$? These covariances represents an extra margin of action for Q. Result 5B analyzes the impact of Q on $\mathcal{R}_0$.

Result 5B. The Sovereign Wealth Fund and the value of the Fiscal Space. *The SWF* *reduces the public debt risk premium*

$$\frac{d\mathcal{R}_0}{dQ} < 0$$

if i) $\alpha < \frac{2}{1-4Q} > 0$ *and ii)*$R > \kappa(Q)$.

Proof. Appendix 4.

Condition i) puts an upper limit on the tax distortion parameter and the size of the SWF purchases ($Q < 1/4$) and ensures $\frac{d}{dQ}\operatorname{Cov}(M^*, T_1^*) > 0$. Besides, generally,

$$\frac{d}{dQ}\operatorname{Cov}(M^*, (r^s)^*) < 0, \quad \frac{d}{dQ}\operatorname{Cov}(M^*, G_1^*) < 0.$$

The intuition for the covariance signs is as follows: when Q is implemented, good times become even better with higher consumption and lower marginal utility; while spending becomes even higher (more procyclical), taxes do not fall as much (less procyclical), both increasing $\mathcal{R}_0$. Contrarily, returns get even higher in times of lower marginal utility, bringing the covariance down and reducing $\mathcal{R}_0$. Condition ii), where

$$\kappa(Q) = \frac{1}{\mu_Y}\left(\frac{1+\alpha Q}{\alpha} + Q\right)\left(\mu_Y - \gamma\,\frac{1+\alpha Q}{2+\alpha}\,\sigma_Y^2\right)^2$$

ensures the effect of taxes and spending dominates that on returns. Thus, under the stated conditions, Q reduces the bond risk premium. Even if readers are uncomfortable with the conditions, they can at least acknowledge that the effect of Q is ambiguous—a remarkable finding given the absorption of risk in the balance sheet. Altogether, the SWF reduces surplus riskiness by generating higher surpluses during periods of high marginal utility. While the government always insures bondholders (maintaining zero-beta debt), the SWF increases taxpayers' insurance, though at the cost of more procyclical spending.

6.3. Comparison with the Debt Revenue

We now derive the so-called Debt Revenue (DR) (Reis (2021), Reis (2022)) in the two-period model. DR represents the implicit transfer that the Treasury obtains by virtue of its ability to raise capital at a lower cost than the private sector. In particular, if future deficits are discounted at the risky return rather than the risk-free rate, an additional term appears in the IBC:

$$
\begin{aligned}
\mathcal{B}^g_{-1} &= \mathcal{S}_0 + \mathbb{E}_0\left[\frac{1+g}{1+r^s}\mathcal{S}_1\right] + \mathbb{E}_0\left[\left(\frac{1}{1+r} - \frac{1}{1+r^s}\right)(1+g)\mathcal{S}_1\right] \\
&\approx \underbrace{\mathcal{S}_0 + \mathbb{E}_0\left[\frac{1+g}{1+r^s}\mathcal{S}_1\right]}_{\text{EPV Surpluses Discounted with } r^s} + \underbrace{\mathbb{E}_0\left[(r^s - r)(1+g)\mathcal{B}^g\right]}_{\text{Debt Revenue}},
\end{aligned}
\tag{28}
$$

where the approximation holds for sufficiently small values of r. In this formulation, DR is interpreted as an implicit revenue: if $r < r^s$, then Treasury bonds are valued more highly than private assets, or equivalently, the Treasury's cost of capital is lower than that of private agents. This representation constitutes an alternative formulation of the IBC, one that implicitly argues that the "correct" borrowing cost for the Treasury should be r^s.

The choice of r^s as the relevant discount factor is unclear, though. In our model, $r^s > r$ (on expectation) does not emerge from bubbles, convenience yields or repression, but from standard asset pricing, related to the different risk profiles of the assets. In contrast to the DR, the SWF operation delivers explicit revenues to the Treasury, coming from the absorption of risk in exchange for a risk premium.

7. The Optimal Size of the Sovereign Wealth Fund

If SWF managers had a good understanding of the macroeconomy, how much would they invest in risky assets? In other words, how big should the SWF balance sheet be? A way of answering this question is to solve the following problem:

$$
Q^* = arg \max_{Q \leq \bar{Q}} U(C_0(Q), G_0(Q)) + \delta \mathbb{E}_0[U(C_1(Q), G_1(Q))]
\tag{29}
$$

given the competitive equilibrium characterized in Section 4. The following result states the solution.

Result 7. The optimal size of the Sovereign Wealth Fund. *In the economy with exogenous R, the optimal amount of risky assets acquired by the fund is given by $Q^* = \bar{Q}$.*

Proof. Appendix 5.

Result 7 establishes that, in an economy with exogenous interest rate R, the optimal Q reaches

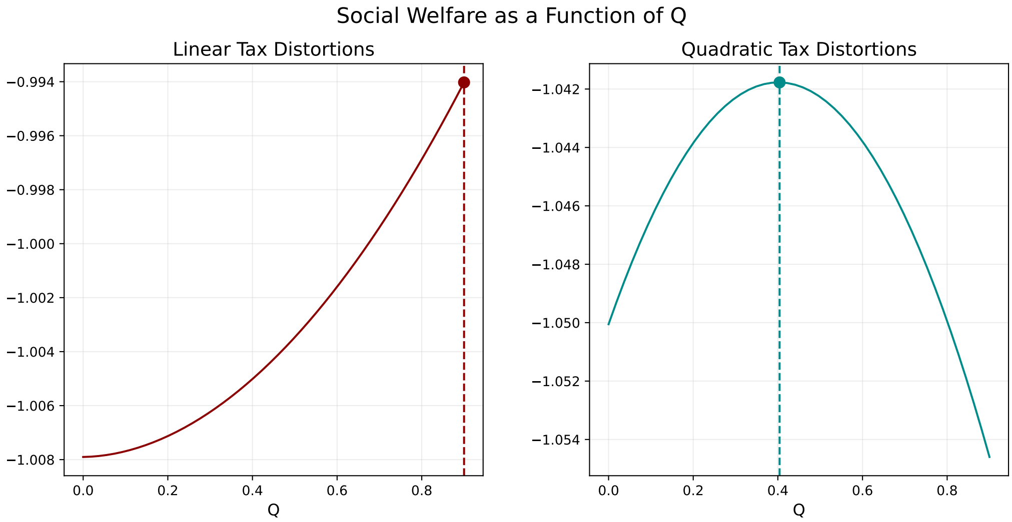

Figure 2: Q^* ***with endogenous*** $\boldsymbol{R}$ ***and linear and quadratic tax distortions***

its maximum permissible value, $Q^* = \bar{Q}$. This finding emerges from the favorable trade-off between risk and return that SWF investment creates for the economy. As the SWF accumulates risky assets, it generates a positive effect on both the mean and variance of future consumption and government spending. The proof shows that the mean increases more rapidly than the variance, improving the certainty equivalent welfare. This result is similar to Aiyagari et al. (2002), but in a context of aggregate risk: the government should accumulate as much assets as possible, to reduce taxes as much as possible.

As figure 2 illustrates, the result $Q^* = \bar{Q}$ emerges also for the case with the endogenous R, for the same reason: a higher Q is monotonically related to lower distortions and higher output, consumption, and spending. Instead, the right panel plot the welfare function with a quadratic tax function $H(T) = \alpha T^2$, capturing the possibility that tax collection is increasingly costly, or the fact that higher taxes are increasingly damaging for the economy, desincentivizing agents to work or to save. The key factor is the marginal productivity of tax cuts. With linear costs, it was constant; every additional unit of Q generates the same efficiency gain, but with quadratic costs, the efficiency gains are decreasing on Q such that at some point they are easily outweighed by the risks associated to higher Q. Thus, quadratic tax distortions give rise to an interior solution.

Our findings stand in contrast to Farhi (2010)'s analysis, which argues that productivity shocks necessitate a short position in risky assets. The intuition behind this recommendation is rooted in the negative correlation between productivity shocks and fiscal needs. When adverse productivity shocks occur, tax revenues decline while fiscal requirements typically increase, creating budget pressure. Farhi argues that this natural fiscal exposure can be hedged through a short position in capital. In contrast, in our framework, when a negative output shock occurs, government spending adjusts downward, which reduces funding requirements and allows for smoother tax rates. Although holding capital exposes the government to risk, endogenous spending serves as an absorption mechanism that limits this exposure. Then, efficiency gains from reduced tax distortions can dominate the costs of bearing risk, making it optimal for the government to hold a long position in capital—in fact, with linear distortions, to hold the maximum permissible amount. This fundamental difference in results highlights the critical role of spending flexibility as a fiscal adjustment channel that transforms the government's optimal portfolio strategy.

8. Conclusions

This paper analyzes the macroeconomic and asset pricing implications of a leveraged SWF that issues risk-free liabilities to purchase domestic risky assets. We show that a SWF capturing the equity premium expands fiscal space, allowing for tax cuts that reduce distortions and stimulate output, consumption, and government spending. However, this comes at the cost of introducing additional risk onto the public balance sheet, leading to more procyclical government spending and private consumption, which deteriorate the hedging properties of stocks. This deterioration increases the equity premium and triggers a flight-to-safety effect that increases demand for government bonds, ultimately lowering the risk-free rate despite additional bond issuance. For debt sustainability, the SWF introduces a fourth determinant beyond conventional factors—namely, the capital income derived from operating the fund. With linear tax distortions, optimal policy funds the SWF to its maximum feasible size; with quadratic distortions, the optimum balances efficiency gains against increased volatility.

While our assumptions yield a sharp characterization of the efficiency-risk trade-off facing a SWF, they raise questions about the generality of our results. Further research exploring alternative welfare functions, shock distributions, and timing protocols—particularly models with full government commitment—is necessary to determine the robustness of our findings.

Moving our framework to a fully dynamic setup is a necessary extension. This would enrich the analysis by transforming our sequential game into a setting with strategic considerations about fiscal credibility, learning, and time-varying risk premia. Such an extension would illuminate how SWF operations might optimally respond to business cycles. Additionally, complementing our risk-oriented SWF with a stabilization fund would implement government-level precautionary savings to smooth spending across states and time. This dual-fund structure could address procyclicality concerns while preserving efficiency benefits.

A third domain for extension is empirical analysis. While the SWF we describe is largely an abstract construct that departs from existing SWFs, the analogy with certain versions of Quantitative Easing provides historical data to test hypotheses developed in the paper. We are currently working on all these fronts.

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Appendix

1. Proof of Result 1

Consider the government budget constraint

$$B^g = T_1 - G_1 + X, \tag{30}$$

where B^g is predetermined, and X denotes state–contingent transfers. Taking covariances with aggregate output Y_1 and defining the output betas as

$$\beta_{T_1} \equiv \frac{\mathrm{Cov}(T_1, Y_1)}{\mathrm{Var}(Y_1)}, \quad \beta_{G_1} \equiv \frac{\mathrm{Cov}(G_1, Y_1)}{\mathrm{Var}(Y_1)}, \quad \beta_X \equiv \frac{\mathrm{Cov}(X, Y_1)}{\mathrm{Var}(Y_1)},$$

the fact that B^g is non-stochastic implies

$$\mathrm{Cov}(B^g, Y_1) = 0 = \mathrm{Cov}(T_1 - G_1 + X, Y_1),$$

or equivalently,

$$\beta_{T_1} - \beta_{G_1} + \beta_X = 0. \tag{31}$$

Note that the optimal fiscal rules and transfers are affine in output:

$$X = Q\,Y_1 + (\text{constant}).$$

and

$$T_1^* = \frac{1-2Q}{2+\alpha}\,Y_1 + (\text{constant}), \quad G_1^* = \frac{1+\alpha Q}{2+\alpha}\,Y_1 + (\text{constant}).$$

Thus, the closed–form expressions for the output betas are

$$\beta_{T_1} = \frac{1-2Q}{2+\alpha}, \quad \beta_{G_1} = \frac{1+\alpha Q}{2+\alpha}.$$

Since $X = Q\,Y_1 + (\text{constant})$, we have

$$\beta_X = Q.$$

These betas satisfy:

$$\beta_{T_1} - \beta_{G_1} = \frac{1-2Q}{2+\alpha} - \frac{1+\alpha Q}{2+\alpha} = -\frac{(2+\alpha)Q}{2+\alpha} = -Q = -\beta_X.$$

The derivate of the betas with respect to Q gives the expressions in the result. Note $\beta_{G_1} = \beta_{C_1}$.

Now, we show the relationship between these output-betas and risk-betas. Let the pricing kernel be given by

$$M = \exp\Big(-\gamma\big(C_1^* - C_0\big)\Big),$$

and define the *risk beta* of any instrument Z as

$$\beta_Z^M \equiv -\frac{\mathrm{Cov}(M, Z)}{\mathrm{Var}(M)},$$

as opposed to the *output beta*

$$\beta_Z^Y \equiv \frac{\mathrm{Cov}(Z, Y_1)}{\mathrm{Var}(Y_1)}.$$

Then, under the assumption that C_1^* is affine in Y_1, the risk betas are equivalent to the output betas. Because C_1^* is affine in Y_1, there exist constants c_0 and c_1 such that

$$C_1^* = c_0 + c_1 Y_1 \quad \text{with} \quad c_1 = \frac{1+\alpha Q}{2+\alpha}.$$

Thus, the pricing kernel can be rewritten as

$$M = \exp\Big(-\gamma(c_0 - C_0)\Big)\exp\Big(-\gamma c_1 Y_1\Big) = A\,\exp\Big(-\gamma c_1\,Y_1\Big),$$

where $A = \exp\big(-\gamma(c_0 - C_0)\big)$ is a positive constant. Hence, M is a strictly monotonic function of Y_1. For small fluctuations in Y_1 (or under the assumption of log-normality), we can approximate M via a first-order Taylor expansion about $\mathbb{E}[Y_1]$:

$$M \approx A\,\exp\Big(-\gamma c_1\,\mathbb{E}[Y_1]\Big)\,[1 - \gamma c_1\,(Y_1 - \mathbb{E}[Y_1])] \equiv K\Big[1 - \gamma c_1\,(Y_1 - \mathbb{E}[Y_1])\Big],$$

where $K = A\,\exp\Big(-\gamma c_1\,\mathbb{E}[Y_1]\Big)$. Then, for any instrument Z (affine in Y_1), we have

$$\mathrm{Cov}(M, Z) \approx -K\gamma c_1\;\mathrm{Cov}(Y_1, Z),$$

and

$$\mathrm{Var}(M) \approx K^2(\gamma c_1)^2\;\mathrm{Var}(Y_1).$$

Thus, the risk beta of Z is given by

$$\beta_Z^M \equiv -\frac{\mathrm{Cov}(M, Z)}{\mathrm{Var}(M)} \approx \frac{\mathrm{Cov}(Y_1, Z)}{K\,\gamma c_1\;\mathrm{Var}(Y_1)}.$$

Since $K\,\gamma c_1$ is a constant common to all instruments, it follows (after appropriate normalization of the pricing kernel) that

$$\beta_Z^M \propto \frac{\mathrm{Cov}(Y_1, Z)}{\mathrm{Var}(Y_1)} = \beta_Z^Y.$$

If we choose the normalization such that the proportionality factor equals one, we obtain the equivalence

$$\beta_Z^M = \beta_Z^Y.$$

2. Proof of Result 2

We start by deriving the equilibrium pricing function. Recall the FOCs for the private sector. For bonds:

$$exp\{-\gamma C_0\} = \beta R\mathbb{E}_0\Big[exp\{-\gamma C_1^*\}\Big] \tag{32}$$

and for stocks:

$$exp\{-\gamma C_0\} = \beta\frac{1}{P}\mathbb{E}_0\Big[exp\{-\gamma C_1^*\}Y_1\Big] \tag{33}$$

Plugging the first into the second and isolating prices on the LHS:

$$P = \frac{1}{R}\frac{\mathbb{E}_0\Big[exp\{-\gamma C_1^*\}Y_1\Big]}{\mathbb{E}_0\Big[exp\{-\gamma C_1^*\}\Big]} \tag{34}$$

Since C_1^* is affine in output, following the notation of the Proof of Result 1,

$$\mathbb{E}_0\Big[exp\{-\gamma C_1^*\}Y_1\Big] = \mathbb{E}_0\Big[exp\{-\gamma\frac{1+\alpha Q}{2+\alpha}Y_1 - \gamma c_0\}Y_1\Big] = exp\{-\gamma c_0\}\mathbb{E}_0\Big[exp\{yY_1\}Y_1\Big] \tag{35}$$

for $y \equiv -\gamma\frac{1+\alpha Q}{2+\alpha}$. Using the properties of Normal and log-Normal distributions:

$$\mathbb{E}_0\Big[exp\{yY_1\}Y_1\Big] = exp\{y\mu_Y + \frac{y^2}{2}\sigma_Y^2\}(\mu_Y + y\sigma_Y^2) \tag{36}$$

Besides,

$$\mathbb{E}_0\Big[exp\{-\gamma C_1^*\}\Big] = exp\{-\gamma c\}\mathbb{E}_0\Big[exp\{yY_1\}\Big] = exp\{-\gamma c_0\}exp\{y\mu_Y + \frac{y^2}{2}\sigma_Y^2\} \tag{37}$$

Therefore,

$$P^* = \frac{1}{R}\Big(\mu_Y - \gamma\frac{1+\alpha Q}{2+\alpha}\sigma_Y^2\Big) \tag{38}$$

It is immediate to see that it is decreasing on Q, as all the parameters are non-negative.

Then, we need to show that an increase in the parameter Q raises the risk premium, which is given by

$$\mathbb{E}_0\Big(\frac{Y_1}{P^*} - R\Big) = \frac{\mu_Y R}{\mu_Y - \gamma\Big(\frac{1+\alpha Q}{2+\alpha}\Big)\sigma_Y^2} - R$$

It is immediate to see that it is increasing in Q.

Finally, we derive the asset demands. The stock market clearing condition implies $S(Q,P)+Q = 1$. Solving that equation for P should deliver the pricing function (18). A function that accomplishes this is

$$S^d = S(Q,R,P) = \frac{\mu_Y - RP}{\gamma\sigma_Y^2} - \frac{1+\alpha}{2+\alpha}(2Q-1) \tag{39}$$

The first element of the function is the well-known stock demand in CARA-Normal environments. The second element is related to the presence of fiscal frictions and Q. The partial derivative of that function with respect to Q delivers the expression stated in the result.

Using the stock demand (39) and the consumption demand (13) in the private budget constraint,

$$\frac{B^d}{R} = (P + Y_0)S_{-1} - T_0 - H(T_0) - C_0^d - PS^d \tag{40}$$

Hence, from the private agent standpoint,

$$
\begin{aligned}
\frac{\partial B^d/R}{\partial Q} &= -\frac{dC_0^d}{dQ} - P\frac{dS^d}{dQ} \\
&= \frac{\alpha(\mu_Y - PR)}{(2+\alpha)} - \frac{\alpha\gamma\sigma_Y^2(1+\alpha Q)}{(2+\alpha)^2} + \frac{(2+2\alpha)P}{2+\alpha} > \frac{\partial}{\partial Q}\frac{B^c}{R} = P
\end{aligned}
$$

$\mu_Y - PR \geq \gamma\sigma_Y^2 \frac{1+\alpha Q}{2+\alpha}$, which holds in equilibrium, is sufficient for the inequality to hold.

In the text, we claim that the common force behind these effects was the higher covariance between period 1 consumption and output. Now we formalize that effect on the risk premium, proceeding in several steps.

i) By Result 1, the sensitivity of period 1 consumption C_1 to the aggregate shock Y_1 is given by

$$
\beta_{C_1} \equiv \frac{\mathrm{Cov}_0(C_1, Y_1)}{\sigma_Y^2} = \frac{1+\alpha Q}{2+\alpha}.
$$

such that

$$
\frac{d\beta_{C_1}}{dQ} = \frac{\alpha}{2+\alpha} > 0.
$$

Thus, an increase in Q raises β_{C_1} and, holding σ_Y^2 constant, implies that

$$
\frac{d}{dQ}\mathrm{Cov}_0(C_1, Y_1) = \sigma_Y^2 \frac{d\beta_{C_1}}{dQ} > 0.
$$

ii) Under a CARA utility specification with risk aversion parameter γ, the marginal utility is given by

$$
U'(C_1) = \exp\{-\gamma C_1\}.
$$

Since $U'(C_1)$ is strictly decreasing in C_1, an increase in $\mathrm{Cov}_0(C_1, Y_1)$ implies that the covariance between marginal utility and Y_1 moves in the opposite direction. More precisely,

$$
\frac{d}{dQ}\mathrm{Cov}_0\big(U'(C_1), Y_1\big) < 0.
$$

iii) Note that, at time 0, the asset price P, the risk-free rate R, and the initial consumption C_0 are known. Therefore, we can reexpress the covariance as

$$
\frac{1}{P}\mathrm{Cov}_0\big(U'(C_1), Y_1\big) = \mathrm{Cov}_0\left(\frac{U'(C_1)}{U'(C_0)}, \frac{Y_1}{P} - R\right),
$$

since $U'(C_0)$, P, and R are constant with respect to the uncertainty in period 1 outcomes.

iv) The fundamental asset pricing equation implies that the excess return is given by

$$\mathbb{E}_0\Big(\frac{Y_1}{P} - R\Big) = -R \ \mathrm{Cov}_0\left(\frac{U'(C_1)}{U'(C_0)}, \frac{Y_1}{P} - R\right).$$

Combining this with the result from (iii) and the finding in (ii) that

$$\frac{d}{dQ}\,\mathrm{Cov}_0\big(U'(C_1), Y_1\big) < 0,$$

it follows that an increase in Q leads to a *more negative* covariance between the pricing kernel and the asset payoff. Consequently, the risk premium, given by $\mathbb{E}_0(Y_1/P - R)$, must increase.

3. Proof of Result 5A

The first claim is that fiscal surpluses-to-output are decreasing on Q in both periods. Formally,

$$\frac{d}{dQ}\Big(\frac{T_0^* - G_0^*}{Y_0^*}\Big) = \frac{1}{(Y_0^*)^2}\left[Y_0^*(1+\alpha/2) + \alpha(T_0^* - G_0^*)\right]\frac{dT_0^*}{dQ}$$

Since $\frac{dT_0^*}{dQ} < 0$ by Result 4, $Y_0^*(1+\alpha/2) + \alpha(T_0^* - G_0^*) > 0$ is necessary and sufficient for ensuring the sign of the overall derivative is negative. The condition puts an upper bound on deficits, but it is rather relaxed for real-world deficit levels. The same condition applies for period 1.

The effect of Q on the net output of period 0 also follows from short-run multipliers. The sign of the derivative dg/dQ is the same as the sign of

$$\frac{d\Delta Y^*}{dQ} = \frac{dY_1^*}{dQ} - \frac{dY_0^*}{dQ} = \frac{2\alpha}{2+\alpha}\left(Y_1 - \mu_Y + \gamma\Big(\frac{1+\alpha Q}{2+\alpha}\Big)\sigma_Y^2\right) \tag{41}$$

where the last expression comes from using the short and long-run Q-multipliers, and includes the indirect effect of Q on Y_1^* through B^g and X. The sign of $\frac{d\Delta Y^*}{dQ}$ depends on the realization of Y_1. On expectation, the sign is positive.

The effect of Q on the expected present value of period 1 surpluses is in principle ambiguous. However, the following computation shows that the overall sign is negative, reflecting that the reduction in the surpluses more than offset the higher growth. Given

$$\mathcal{S}_1^* = \frac{T_1^* - G_1^*}{Y_1^*} = \frac{B^g - X}{Y_1^*}$$

Then, the expected discounted surplus is

$$\mathbb{E}_0[(1+g^*)\mathcal{S}_1^*] = \frac{B^g + QRP^* - Q\,E(Y_1)}{Y_0^*}\,,$$

The quotient rule yields

$$\frac{d}{dQ}\mathbb{E}_0[(1+g^*)\mathcal{S}_1^*] = \frac{1}{Y_0^*}\left\{\frac{dB^g}{dQ} + RP^* - \mu_Y - \mathbb{E}_0[(1+g^*)\mathcal{S}_1^*]\frac{dY_0^*}{dQ}\right\}. \tag{42}$$

Using the short-run Q-multipliers and equilibrium prices,

$$\frac{dB^g}{dQ} + RP^* - \mu_Y = -\frac{\gamma\sigma_Y^2}{2+\alpha}\cdot\frac{1+R+\alpha Q}{1+R} = -\frac{\gamma\sigma_Y^2}{2+\alpha}\left(1+\frac{\alpha Q}{1+R}\right).$$

Then,

$$\frac{d}{dQ}\mathbb{E}_0[(1+g^*)\mathcal{S}_1^*] = \frac{1}{Y_0^*}\left\{-\frac{\gamma\sigma_Y^2}{2+\alpha}\left(1+\frac{\alpha Q}{1+R}\right) - \mathbb{E}_0[(1+g^*)\mathcal{S}_1^*]\frac{dY_0^*}{dQ}\right\} < 0 \tag{43}$$

Since both terms inside the braces are negative (noting that $\frac{dY_0^*}{dQ} > 0$), this expression has a negative sign.

One of the determinants of the expected present value of deficits is $\mathrm{Cov}_0(1+g^*, \mathcal{S}_1^*)$. A similar argument yields

$$\mathrm{Cov}_0(1+g^*, \mathcal{S}_1^*) = \frac{B^g + QP^*R}{Y_0^*}\left[1 - E(Y_1^*)\,E\Big(\frac{1}{Y_1^*}\Big)\right]$$

Taking the total derivative with respect to Q, acknowledging the indirect effects through debt, output and stock prices,

$$\frac{d}{dQ}\,\mathrm{Cov}_0(1+g^*, \mathcal{S}_1^*) = \left[1 - E(Y_1^*)\,E\left(\frac{1}{Y_1^*}\right)\right]\cdot\frac{\left(P^*R - \frac{Q\,\alpha\gamma\sigma_Y^2}{(1+R)(2+\alpha)}\right)Y_0^* - (B^g + QP^*R)\,\frac{2\,Q\,\alpha^2\gamma\sigma_Y^2}{(1+R)(2+\alpha)^2}}{(Y_0^*)^2}$$

By Jensen's inequality,

$$E(Y_1^*)\,E\left(\frac{1}{Y_1^*}\right) > 1,$$

so that the square bracket is negative. Note that

$$\left(P^*R - \frac{Q\,\alpha\gamma\sigma_Y^2}{(1+R)(2+\alpha)}\right) = \mu_Y - \gamma\Big(\frac{1+\alpha Q}{2+\alpha}\Big)\sigma_Y^2 - \frac{Q\,\alpha\gamma\sigma_Y^2}{(1+R)(2+\alpha)}$$

and the second element in the numerator of the fraction is negative. Therefore, the numerator is negative for reasonable values for μ_Y and $\frac{d}{dQ}\,\mathrm{Cov}_0(1+g^*, \mathcal{S}_1^*) < 0$.

Now, we want to show

$$\frac{d\frac{(\mathcal{B}^c)^*}{1+r}\mathbb{E}_0[(r^s)^* - r]}{dQ} > 0$$

The derivative of $(\mathcal{B}^c)^*$ with respect to Q (by the quotient rule) is

$$\frac{d(\mathcal{B}^c)^*}{dQ} = \frac{1}{Y_0^*}\frac{d(B^c)^*}{dQ} - \frac{(B^c)^*}{(Y_0^*)^2}\frac{dY_0^*}{dQ} = \frac{P^*R}{Y_0^*} - \frac{(\mathcal{B}^c)^*}{Y_0^*}\frac{dY_0^*}{dQ}.$$

Differentiating $F(Q)$ with respect to Q (noting that r and $1+r$ are constant) yields

$$\frac{dF}{dQ} = \frac{1}{1+r}\left\{\frac{d(\mathcal{B}^c)^*}{dQ}\left[\frac{\mathbb{E}_0[Y_1]}{P^*} - 1 - r\right] + (\mathcal{B}^c)^*\frac{d}{dQ}\left(\frac{\mathbb{E}_0[Y_1]}{P^*}\right)\right\}.$$

Since $\mathbb{E}_0[Y_1]$ does not depend on Q, we have

$$\frac{d}{dQ}\left(\frac{\mathbb{E}_0[Y_1]}{P^*}\right) = -\frac{\mathbb{E}_0[Y_1]}{(P^*)^2}\frac{dP^*}{dQ} = \frac{\gamma\,\alpha\,\sigma_Y^2\,\mathbb{E}_0[Y_1]}{(2+\alpha)R\,(P^*)^2}$$

Substituting the expressions into the derivative of $F(Q)$ gives

$$\frac{dF}{dQ} = \frac{1}{1+r}\left\{\left[\frac{P^*R}{Y_0^*} - \frac{(B^c)^*}{Y_0^*}\frac{dY_0^*}{dQ}\right]\left[\frac{\mathbb{E}_0[Y_1]}{P^*} - 1 - r\right] + (\mathcal{B}^c)^*\frac{\gamma\,\alpha\,\sigma_Y^2\,\mathbb{E}_0[Y_1]}{(2+\alpha)R\,(P^*)^2}\right\}.$$

Using the SWF budget constraint,

$$\left[\frac{P^*R}{Y_0^*} - \frac{(B^c)^*}{Y_0^*}\frac{dY_0^*}{dQ}\right] = \frac{1}{Y_0^*}\left[P^*R\Big(1 - \frac{2\alpha^2\gamma\sigma_Y^2\,Q^2}{(1+R)(2+\alpha)^2}\Big)\right]$$

which is larger than 0 if $\left(1 - \frac{2\alpha^2\gamma\sigma_Y^2\,Q^2}{(1+R)(2+\alpha)^2}\right) > 0$.

4. Proof of Result 5B

We derive conditions under which

$$\frac{d\mathcal{R}_0}{dQ} = \frac{d}{dQ}\,\mathrm{Cov}(M^*, T_1^*) - \frac{d}{dQ}\,\mathrm{Cov}(M^*, G_1^*) + \frac{d}{dQ}\,\mathrm{Cov}(M^*, (r^s)^*) < 0$$

where the covariances are conditional on period 0 information throughout the proof. In our model, all period-1 variables are affine in Y_1 and the pricing kernel takes the exponential–affine form

$$M^* = A\,e^{-\lambda Y_1}, \quad \text{with} \quad A = \delta\,e^{-\gamma K}, \quad \lambda = \gamma\,\frac{1+\alpha Q}{2+\alpha}\,.$$

Similarly, any variable Z^* is written as

$$Z^* = c_Z + d_Z\,Y_1.$$

Then one may show that

$$\mathrm{Cov}(M^*, Z^*) = -A\,d_Z\,\lambda\,\sigma_Y^2\,\Phi, \quad \Phi = \exp\Big\{-\lambda\mu_Y + \frac{1}{2}\lambda^2\sigma_Y^2\Big\}\,.$$

Since A, λ, σ_Y^2, and Φ are positive, the sign of $\mathrm{Cov}(M^*, Z^*)$ is determined by $-d_Z$. In our application the slopes are

$$d_T = \frac{1-2Q}{2+\alpha}, \quad d_G = \frac{1+\alpha Q}{2+\alpha}, \quad d_r = \frac{1}{P^*}\,,$$

Under typical parameter values (with Q small enough so that $1 - 2Q > 0$, and with $P^* > 0$) all three slopes are positive. Hence, all three covariances are negative.

Denote generically, for a given Z^*,

$$F_Z(Q) \equiv \mathrm{Cov}(M^*, Z^*) = -A(Q)\,d_Z(Q)\,\lambda(Q)\,\sigma_Y^2\,\Phi(Q).$$

Our goal is to compute

$$\frac{dF_Z}{dQ} = \frac{d}{dQ}\Big[-A\,d_Z\,\lambda\,\sigma_Y^2\,\Phi\Big].$$

Since σ_Y^2 is constant we have

$$\frac{dF_Z}{dQ} = -\sigma_Y^2\,\frac{d}{dQ}\Big[A\,d_Z\,\lambda\,\Phi\Big].$$

Using the product rule,

$$\frac{dF_Z}{dQ} = -\sigma_Y^2 \Big\{ A'(Q)\, d_Z\, \lambda\, \Phi + A(Q)\, d_Z'\, \lambda\, \Phi + A(Q)\, d_Z\, \lambda'(Q)\, \Phi + A(Q)\, d_Z\, \lambda\, \Phi'(Q) \Big\}.$$

By definition,

$$\lambda(Q) = \gamma\, \frac{1+\alpha Q}{2+\alpha} \quad \Longrightarrow \quad \lambda'(Q) = \frac{\gamma\, \alpha}{2+\alpha}\,.$$

Since

$$\Phi(Q) = \exp\Big\{ -\lambda \mu_Y + \tfrac{1}{2}\lambda^2 \sigma_Y^2 \Big\},$$

its derivative is obtained via the chain rule:

$$\Phi'(Q) = \Phi(Q)\Big[-\mu_Y + \lambda\, \sigma_Y^2 \Big] \lambda'(Q).$$

The derivatives of d_Z are given by

$$d_T(Q) = \frac{1-2Q}{2+\alpha} \quad \Longrightarrow \quad d_T'(Q) = -\frac{2}{2+\alpha}\,,$$

$$d_G(Q) = \frac{1+\alpha Q}{2+\alpha} \quad \Longrightarrow \quad d_G'(Q) = \frac{\alpha}{2+\alpha}\,.$$

$$d_r(Q) = \frac{1}{P^*(Q)} \quad \Longrightarrow \quad d_r'(Q) = \frac{\gamma \alpha \sigma_Y^2}{R(2+\alpha)\big(P^*\big)^2}$$

Since

$$A(Q) = \delta\, e^{-\gamma K(Q)},$$

with

$$K(Q) = \frac{(1+\alpha)a - \alpha B^g - \alpha Q\, R\, P^*(Q) - (2+\alpha) C_0^*}{2+\alpha}.$$

The derivative of $K(Q)$ is given by

$$\frac{dK}{dQ} = \frac{-\alpha\, \frac{d(B^g)^*}{dQ} - \alpha R P^* - \alpha Q R\, \frac{dP^*}{dQ} - (2+\alpha)\, \frac{dC_0^*}{dQ}}{2+\alpha} = -\frac{\alpha R\, P^*}{2+\alpha}$$

This result shows that when we account for the contributions from the derivatives of $(B^g)^*$ and C_0^*, they cancel the corresponding derivative coming from the term $-\alpha Q R P^*$ (via dP^*/dQ). The net effect is that the only remaining Q–sensitivity of K comes from the explicit factor $-\alpha R P^*$, leading to the simple closed–form derivative above. Then,

$$A'(Q) = -\gamma\, A(Q)\, K'(Q) = \gamma\, A(Q) \frac{\alpha R\, P^*}{2+\alpha}$$

Collecting the above derivatives, one can show that the derivative $\frac{dF_Z}{dQ}$ may be expressed in a compact form where its sign is determined by a linear combination of the terms involving d_Z and d'_Z. In particular,

$$\frac{dF_Z}{dQ} = -\sigma_Y^2\, A\, \lambda^2\, \Phi\, \left\{ d_Z\, \frac{\gamma\alpha}{2+\alpha} \Big(RP^* + \Delta \Big) + d'_Z \right\}.$$

where

$$\Delta \equiv \frac{2+\alpha}{\gamma(1+\alpha Q)} - \mu_Y + \gamma\, \frac{1+\alpha Q}{2+\alpha}\, \sigma_Y^2\, .$$

Since the multiplicative factor $-\sigma_Y^2\, A\, \lambda^2\, \Phi$ is negative, the sign of $\frac{dF_Z}{dQ}$ is opposite to that of the bracketed term

$$\Xi_Z \equiv d_Z\, \frac{\gamma\alpha}{2+\alpha} \Big(RP^* + \Delta \Big) + d'_Z.$$

For T_1^* we require that

$$\frac{d}{dQ}\, \mathrm{Cov}(M^*, Z^*) > 0,$$

which, given the overall negative sign, is equivalent to imposing that

$$\Xi_Z < 0.$$

A sufficient condition to ensure this is that

$$RP^* + \Delta < \frac{2(2+\alpha)}{\gamma\alpha(1-2Q)},$$

after substituting the specific expressions for d_Z and d'_Z (for example, for T_1^* one has $d_T = (1-2Q)/(2+\alpha)$ and $d'_T = -2/(2+\alpha)$).

In other words, when the combined term $RP^* + \Delta$ is small enough—relative to the threshold $\frac{2(2+\alpha)}{\gamma\alpha(1-2Q)}$—the bracketed term Ξ_Z becomes negative, thereby ensuring that the derivative of the covariance is positive (i.e., the negative covariance becomes less negative) for T_1^* and $(r^s)^*$. In contrast, for G_1^* the sign of d_G and its derivative d'_G ensure that the derivative of $\mathrm{Cov}(M^*, G_1^*)$ remains negative, as desired.

Thus, the analysis of $\frac{dF_Z}{dQ}$ reduces to checking the inequality

$$RP^* + \Delta < \frac{2(2+\alpha)}{\gamma\alpha(1-2Q)},$$

Using equilibrium prices,

$$RP^* + \Delta = \Big(\mu_Y - \gamma\,\frac{1+\alpha Q}{2+\alpha}\,\sigma_Y^2\Big) + \Big(\frac{2+\alpha}{\gamma(1+\alpha Q)} - \mu_Y + \gamma\,\frac{1+\alpha Q}{2+\alpha}\,\sigma_Y^2\Big) = \frac{2+\alpha}{\gamma(1+\alpha Q)}.$$

Substituting into the sufficient condition gives

$$\frac{2+\alpha}{\gamma(1+\alpha Q)} < \frac{2(2+\alpha)}{\gamma\alpha(1-2Q)}.$$

Cancelling the positive factor $\frac{2+\alpha}{\gamma}$ from both sides yields

$$\frac{1}{1+\alpha Q} < \frac{2}{\alpha(1-2Q)}.$$

Multiplying through by the positive quantity $\alpha(1-2Q)(1+\alpha Q)$ we obtain

$$\alpha(1-2Q) < 2(1+\alpha Q).$$

Expanding and rearranging:

$$\alpha - 2\alpha Q < 2 + 2\alpha Q \quad\Longrightarrow\quad \alpha < 2 + 4\alpha Q.$$

Subtracting $4\alpha Q$ from both sides gives

$$\alpha(1-4Q) < 2.$$

That is, the desired condition is

$$\alpha < \frac{2}{1-4Q},$$

with the natural restriction $1-4Q > 0$ (i.e., $Q < \frac{1}{4}$). This single condition on α (relative to Q)

ensures that

$$\frac{d}{dQ}\,\mathrm{Cov}(M^*, T_1^*) > 0$$

For government spending G_1^* we have

$$d_G(Q) = \frac{1+\alpha Q}{2+\alpha}, \quad d_G'(Q) = \frac{\alpha}{2+\alpha},$$

so that $\Xi_G > 0$ and

$$\frac{d}{dQ}\,\mathrm{Cov}(M^*, G_1^*) < 0.$$

(Its (negative) covariance becomes more negative.)

For the risky asset $(r^s)^*$ we assume the affine representation

$$d_r(Q) = \frac{1}{P^*(Q)}, \quad d_r'(Q) = \frac{\gamma\alpha\,\sigma_Y^2}{R(2+\alpha)\big(P^*(Q)\big)^2}.$$

Then the corresponding quantity is

$$\Xi_r = \frac{1}{P^*(Q)}\,\frac{\gamma\alpha}{2+\alpha}\Big(RP^* + \Delta\Big) + \frac{\gamma\alpha\,\sigma_Y^2}{R(2+\alpha)\big(P^*(Q)\big)^2}.$$

Since all terms are positive, $\Xi_r > 0$ and hence

$$\frac{d}{dQ}\,\mathrm{Cov}(M^*, (r^s)^*) < 0.$$

In terms of the overall bond risk premium:

$$\frac{d\mathcal{R}_0}{dQ} = \frac{d}{dQ}\,\mathrm{Cov}(M^*, T_1^*) - \frac{d}{dQ}\,\mathrm{Cov}(M^*, G_1^*) + \frac{d}{dQ}\,\mathrm{Cov}(M^*, (r^s)^*).$$

Inserting the derivative expressions we obtain

$$\frac{d\mathcal{R}_0}{dQ} = -\sigma_Y^2\,A\,\lambda\,\Phi\Big\{\Xi_T - \Xi_G + \Xi_r\Big\}.$$

Since the prefactor $-\sigma_Y^2\,A\,\lambda\,\Phi$ is negative, the sign of $\frac{d\mathcal{R}_0}{dQ}$ is the opposite of the sign of

$$\Xi_T - \Xi_G + \Xi_r.$$

Therefore, to ensure that $\frac{d\mathcal{R}_0}{dQ} < 0$ (i.e. that the overall risk measure declines with Q), it suffices to have

$$\Xi_T - \Xi_G + \Xi_r > 0.$$

Let us now write these explicitly. For T_1^*:

$$\Xi_T = \frac{1-2Q}{2+\alpha}\,\frac{\gamma\alpha}{2+\alpha}(RP^* + \Delta) - \frac{2}{2+\alpha}.$$

For G_1^*:

$$\Xi_G = \frac{1+\alpha Q}{2+\alpha}\,\frac{\gamma\alpha}{2+\alpha}(RP^* + \Delta) + \frac{\alpha}{2+\alpha}.$$

And for $(r^s)^*$:

$$\Xi_r = \frac{1}{P^*(Q)}\,\frac{\gamma\alpha}{2+\alpha}(RP^* + \Delta) + \frac{\gamma\alpha\,\sigma_Y^2}{R(2+\alpha)\big(P^*(Q)\big)^2}.$$

Then the net condition is

$$\Xi_T - \Xi_G + \Xi_r > 0.$$

A few algebraic manipulations yield (after multiplying through by the positive factor $(2+\alpha)$):

$$\frac{\gamma\alpha}{2+\alpha}(RP^* + \Delta)\Big[(1-2Q) - (1+\alpha Q)\Big] - 2 - \alpha + (2+\alpha)\,\Xi_r > 0.$$

Noting that

$$(1-2Q) - (1+\alpha Q) = -Q(2+\alpha),$$

this condition becomes

$$-\gamma\alpha\,Q\,(RP^* + \Delta) - (2+\alpha) + (2+\alpha)\,\Xi_r > 0.$$

Dividing by $2+\alpha$ (which is positive) gives

$$-\frac{\gamma\alpha\,Q}{2+\alpha}(RP^* + \Delta) - 1 + \Xi_r > 0.$$

That is,

$$\Xi_r > 1 + \frac{\gamma\alpha\,Q}{2+\alpha}(RP^* + \Delta).$$

Recalling the expression for Ξ_r, this inequality can be written explicitly as

$$\frac{\gamma\alpha}{2+\alpha}\frac{RP^* + \Delta}{P^*(Q)} + \frac{\gamma\alpha\,\sigma_Y^2}{R(2+\alpha)\big(P^*(Q)\big)^2} > 1 + \frac{\gamma\alpha\,Q}{2+\alpha}(RP^* + \Delta).$$

Multiplying both sides by $\frac{2+\alpha}{\gamma\alpha}$ (which is positive) yields

$$\frac{RP^* + \Delta}{P^*(Q)} + \frac{\sigma_Y^2}{R\big(P^*(Q)\big)^2} > \frac{2+\alpha}{\gamma\alpha} + Q\,(RP^* + \Delta).$$

A quick calculation shows that

$$RP^* + \Delta = \frac{2+\alpha}{\gamma(1+\alpha Q)}.$$

Thus the left-side of the inequality becomes

$$\frac{RP^* + \Delta}{P^*(Q)} + \frac{\sigma_Y^2}{R\big(P^*(Q)\big)^2} = \frac{\frac{2+\alpha}{\gamma(1+\alpha Q)}}{P^*(Q)} + \frac{\sigma_Y^2}{R\big(P^*(Q)\big)^2}.$$

Using the equilibrium pricing formula,

$$\frac{1}{P^*(Q)} = \frac{R}{\mu_Y - \gamma\,\frac{1+\alpha Q}{2+\alpha}\,\sigma_Y^2},$$

and

$$\frac{1}{\big(P^*(Q)\big)^2} = \frac{R^2}{\Big(\mu_Y - \gamma\,\frac{1+\alpha Q}{2+\alpha}\,\sigma_Y^2\Big)^2}.$$

Thus, the left-side becomes

$$\frac{2+\alpha}{\gamma(1+\alpha Q)} \cdot \frac{R}{\mu_Y - \gamma\,\frac{1+\alpha Q}{2+\alpha}\,\sigma_Y^2} + \frac{\sigma_Y^2}{R} \cdot \frac{R^2}{\Big(\mu_Y - \gamma\,\frac{1+\alpha Q}{2+\alpha}\,\sigma_Y^2\Big)^2}.$$

That is,

$$\frac{R(2+\alpha)}{\gamma(1+\alpha Q)\,U} + \frac{R\,\sigma_Y^2}{U^2},$$

where we have set

$$U \equiv \mu_Y - \gamma\,\frac{1+\alpha Q}{2+\alpha}\,\sigma_Y^2.$$

The right-side of the inequality is

$$\frac{2+\alpha}{\gamma\alpha} + Q\,(RP^* + \Delta) = \frac{2+\alpha}{\gamma\alpha} + Q\,\frac{2+\alpha}{\gamma(1+\alpha Q)} = \frac{2+\alpha}{\gamma}\left(\frac{1}{\alpha} + \frac{Q}{1+\alpha Q}\right).$$

Thus the inequality becomes

$$\frac{R(2+\alpha)}{\gamma(1+\alpha Q)\,M} + \frac{R\,\sigma_Y^2}{M^2} > \frac{2+\alpha}{\gamma}\left(\frac{1}{\alpha} + \frac{Q}{1+\alpha Q}\right).$$

Cancel the common positive factor $\frac{2+\alpha}{\gamma}$ by multiplying both sides by $\frac{\gamma}{2+\alpha}$:

$$\frac{R}{(1+\alpha Q)\,M} + \frac{R\,\gamma\,\sigma_Y^2}{(2+\alpha)\,M^2} > \frac{1}{\alpha} + \frac{Q}{1+\alpha Q}.$$

Next, multiply both sides by $(1+\alpha Q)U^2$ (which is positive) to clear denominators:

$$R\,U + \frac{R\,\gamma\,\sigma_Y^2(1+\alpha Q)}{2+\alpha} > \left(\frac{1+\alpha Q}{\alpha} + Q\right)U^2.$$

Notice that the left-side simplifies neatly:

$$R\,U + \frac{R\,\gamma\,\sigma_Y^2(1+\alpha Q)}{2+\alpha} = R\left[\mu_Y - \gamma\,\frac{1+\alpha Q}{2+\alpha}\,\sigma_Y^2\right] + \frac{R\,\gamma\,\sigma_Y^2(1+\alpha Q)}{2+\alpha} = R\,\mu_Y.$$

Thus, the condition reduces to

$$R\,\mu_Y > \left(\frac{1+\alpha Q}{\alpha} + Q\right)U^2,$$

or, recalling the definition of U,

$$R > \frac{1}{\mu_Y}\left(\frac{1+\alpha Q}{\alpha} + Q\right)\left(\mu_Y - \gamma\,\frac{1+\alpha Q}{2+\alpha}\,\sigma_Y^2\right)^2 \equiv \kappa(Q)$$

5. Proof of Result 6

We consider the planner's problem of choosing

$$Q^* = \arg\max_{Q\le\bar{Q}}\; U\big(C_0(Q), G_0(Q)\big) + \delta\,\mathbb{E}_0\Big[U\big(C_1(Q), G_1(Q)\big)\Big],$$

Because of the CARA-Normal structure, the continuation utility may be rewritten using the certainty equivalent (CE) of a normally distributed variable:

$$\mathrm{CE}(Z) = \mathbb{E}[Z] - \frac{\gamma}{2}\,\mathrm{Var}(Z).$$

We show that the mean-variance (MV) spread of both C_1 and G_1 is increasing in Q (even accounting for the indirect effects through P^* and $(B^g)^*$). This, combined with the fact that

$$\frac{dC_0^*}{dQ} = \frac{dG_0^*}{dQ} > 0,$$

implies that the overall objective is increasing in Q so that if Q is subject to an upper bound $\bar{Q}$, then the optimum is attained at

$$Q^* = \bar{Q}.$$

Define the constant $A \equiv \frac{1}{2+\alpha}$. Then we can rewrite C_1^* as

$$\begin{aligned} C_1^* &= A\Big[(1+\alpha)a - \alpha(B^g)^* - \alpha Q\,R\,P^* + (1+\alpha Q)Y_1\Big] \\ &= A\Big[(1+\alpha)a - \alpha(B^g)^* - \alpha Q\,R\,P^* + (1+\alpha Q)\mu_Y + (1+\alpha Q)(Y_1-\mu_Y)\Big]. \end{aligned} \tag{44}$$

Thus, the mean

$$\mu_C = A\Big[(1+\alpha)a - \alpha(B^g)^* - \alpha Q\,R\,P^* + (1+\alpha Q)\mu_Y\Big],$$

The coefficient on output is $A(1+\alpha Q)$. The variance of C_1^* is

$$\sigma_C^2 = \beta(Q)^2\sigma_Y^2 = A^2(1+\alpha Q)^2\sigma_Y^2.$$

Hence, the certainty equivalent for C_1^* is given by

$$CE(C_1) = \mu_C - \frac{\gamma}{2}\,A^2(1+\alpha Q)^2\sigma_Y^2. \tag{45}$$

Differentiate the variance

$$\frac{d\sigma_C^2}{dQ} = 2A^2(1+\alpha Q)\alpha\,\sigma_Y^2. \tag{46}$$

and the mean

$$\frac{d\mu_C}{dQ} = A^2\alpha\,\gamma\,\sigma_Y^2\left[1 + 2\alpha Q - \frac{\alpha R\,Q}{1+R}\right] \tag{47}$$

after considering the indirect effects of Q through X and B^g, and simplifying. Combining the derivatives,

$$\frac{dCE}{dQ} = \frac{\alpha^2\,\gamma\,\sigma_Y^2\,Q}{(1+R)(2+\alpha)^2}.$$

Since all parameters (α, γ, σ_Y^2, R) are positive and $Q > 0$, it follows that

$$\frac{dCE}{dQ} > 0.$$

An analogous derivation holds for G_1 since it shares a similar structure. Therefore, the indirect effect of Q (through P^* and $(B^g)^*$) reinforces the direct benefit (given that $dC_0^*/dQ > 0$ and $dG_0^*/dQ > 0$). Consequently, the overall objective function,

$$Q \mapsto U\big(C_0(Q), G_0(Q)\big) + \delta\,\mathbb{E}_0\Big[U\big(C_1(Q), G_1(Q)\big)\Big],$$

is strictly increasing in Q. Hence, when Q is subject to an upper bound $\bar{Q}$, the optimal choice is

$$Q^* = \bar{Q}.$$